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ON THE LENGTH OF LINKED CURVES

BY

MICHAEL EDELSTEIN AND BINYAMIN SCHWARZ

ABSTRACT

If two curves in R^3 are linked and at distance 1 from each other then each is of length 2π at least.

In [1] Gehring posed the following problem: "Suppose that A and B are disjoint linked Jordan curves in R^3 which lie at a distance 1 from each other. Show that the length of A is at least 2π. The corresponding result with a positive absolute constant instead of 2π is due to the proposer." Here the "distance" between A and B is understood to mean $\inf\{\| a - b \| : a \in A, b \in B\}$. To say that A and B are linked means that each of the two fails to be contractible in the complement of the other.

It is the purpose of this note to show that 2π is indeed a lower bound (and since it is realized for a pair of suitably disposed linked circles—the glb). In proving the above essential use will be made of a result of Horn [2, lemma 2] according to which any closed rectifiable curve Γ must be contained in an open hemisphere of the unit sphere S if its length $l(\Gamma)$ is less than 2π and $\Gamma \subset S$.

LEMMA 1. *Suppose $\Gamma \subset R^3$ is a rectifiable closed curve of length $l(\Gamma) < 2\pi$ and $c \in R^3$ is such that* $\operatorname{dist}(c, \Gamma) = \inf\{\| c - x \| : x \in \Gamma\} \geqq 1$. *Then $c \notin \operatorname{co} \Gamma$, the convex hull of Γ.*

PROOF. Let S be the sphere of radius 1 about c. Then no point of Γ is in the interior of S. Let Γ' be the image of Γ under the central projection of $R^3 \sim \{c\}$, from c, onto S. Since arclength is not increased under this projection we have $l(\Gamma') \leqq l(\Gamma) < 2\pi$. Hence, by the above mentioned result of Horn, Γ' is contained in some open hemisphere S' of S. Let H be the plane containing the equatorial circle $\operatorname{cl} S' \sim S'$. We know that $c \in H$ and Γ' lies in one of the two

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open halfspaces determined by H; hence so does Γ. It follows that $c \notin \mathrm{co}\,\Gamma$, as claimed.

LEMMA 2. *If A and B are linked curves then $B \cap \mathrm{co}A \neq \emptyset$.*

PROOF. Suppose, for a contradiction, that $B \cap \mathrm{co}A = \emptyset$. Then, since co A is contractible in itself, A too is contractible in co$A$. Hence A is contractible in $R^3 \sim B$ against the hypothesis that A and B are linked.

PROPOSITION. *If A and B are linked rectifiable Jordan curves in R^3 and*

$$\mathrm{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\} = 1$$

then $l(A)$, the length of A, (and $l(B)$) is at least 2π.

PROOF. Let $c \in B \cap \mathrm{co}\,A$ (which is nonempty by Lemma 2). We clearly have $\mathrm{dist}(c, A) \geqq 1$. Since $c \in \mathrm{co}\,A$ it follows from Lemma 1 that $l(A) \geqq 2\pi$, as claimed.

REMARK. Horn's Lemma also states that if $l(\Gamma) = 2\pi$ and $\Gamma \subset S$ then Γ is contained in a closed hemisphere of S. His proof shows that if in addition Γ is not contained in an open hemisphere then it must be a great circle of S or the union of the two semicircles of such great circles. Based on this fact it is not difficult to show that for a pair of linked closed curves A and B, as in the Proposition, to satisfy $l(A) = l(B) = 2\pi$ both A and B must be circles of radius 1 each passing through the center of the other and lying in orthogonal planes.

REFERENCES

1. F. W. Gehring, Problem 7.22 in *Contributions to Analysis, A Collection of Papers Dedicated to Lipman Bers*, Academic Press, New York and London, 1974.

2. R. A. Horn, *On Fenchel's theorem*, Amer. Math. Monthly **78** (1971), 380–381.

DALHOUSIE UNIVERSITY
HALIFAX, N. S.